APEC Math Review

Part 2 Sets

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Vocabulary

- Set
  - $A = \{US, Columbia, Malawi, China\}$,
  - $\mathbb{R}_+ \equiv \{x|x \geq 0\}$
  - I - Integers
- Element
  - $US \in A$
  - $0 \in \mathbb{R}_+$, $0 \notin \mathbb{R}_{++}$
- Subset
  - $A \subset U = \{all\ countries\ in\ the\ world\}$
  - $\mathbb{R}_+ \subset \mathbb{R}$
- Empty set
  - $\emptyset = \{plant\ with\ black\ flowers\}$

Vocabulary

- Complement: A^c
- Set difference: $A \backslash B$
- Intersection: $A \cap B$
- Union: $A \cup B$

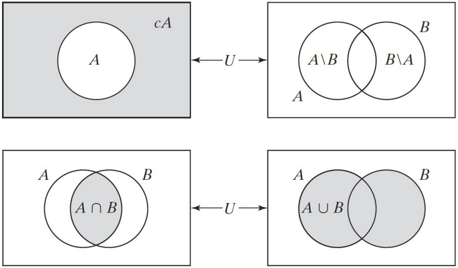


Figure A1. 1. Venn diagrams.

Source: Jehle & Reny (2011)

Vocabulary

Set product - a set of ordered pairs

$$S \times T \equiv \{(s, t) | s \in S,\ t \in T\}$$

N-dimensional Euclidean space

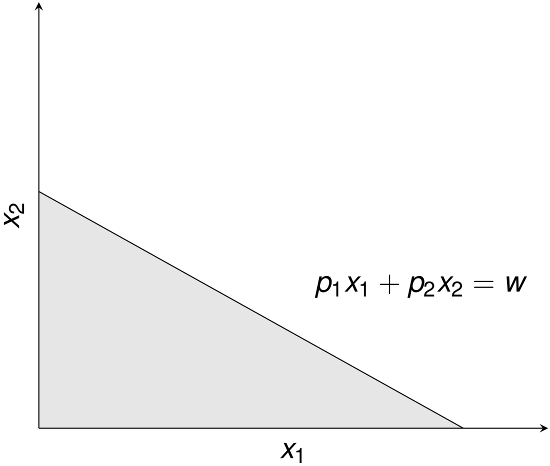
$$\mathbb{R}^n \equiv \mathbb{R} \times \mathbb{R} \times ... \times \mathbb{R} \equiv \{(x_1, ..., x_n) | x_i \in \mathbb{R},\ \forall i = 1, ..., n\}$$

Cartesian Plane

$$\mathbb{R}^2 \equiv \mathbb{R} \times \mathbb{R} \equiv \{(x_1, x_2) | x_1 \in \mathbb{R},\ x_2 \in \mathbb{R}\}$$

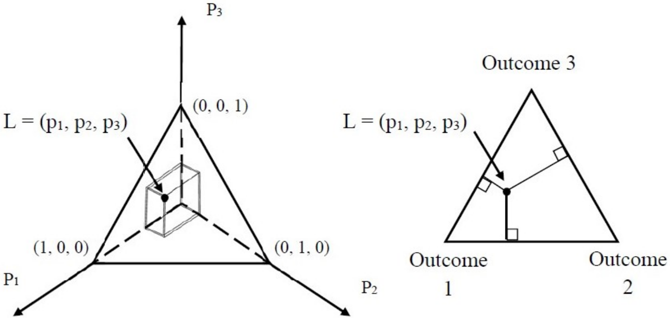
Budget set

$$\{(x_1, x_2)|, x_1 \in [0, \infty), x_2 \in [0, \infty), p_1 x_1 + p_2 x_2 \leq w\}$$



Probability simplex

$$\{(p_1, p_2, p_3) | p_i \in [0, 1] \; for \; i = 1, 2, 3; p_1 + p_2 + p_3 = 1\}$$



Source: Glewwe APEC 8001 lecture notes

Convex set

$S \subset \mathbb{R}^n$ is a convex set of for all $\mathbf{x}^1 \in S$ and $\mathbf{x}^2 \in S$, we have

$$t\mathbf{x}^1 + (1-t)\mathbf{x}^2 \in S$$

for all t in the interval $0 \leq t \leq 1$.

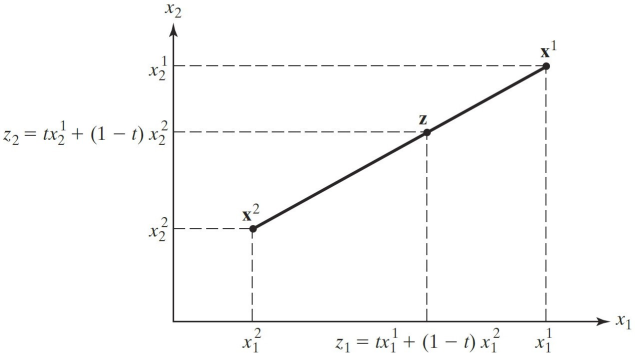


Figure A1. 4. Some convex combinations in $\mathbb{R}^2$.

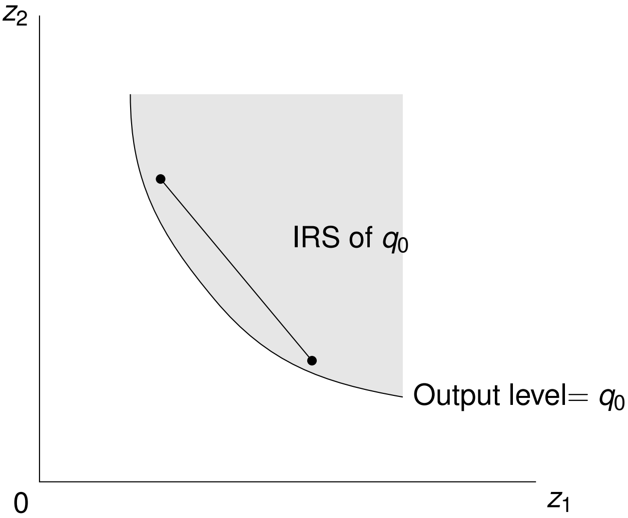
Source: Jehle & Reny (2011)

Convex set

Question: Are these sets convex?

- $\emptyset$
- $\mathbb{R}$
- $S \cup T$ (S and T are convex)
- $S \cap T$ (S and T are convex)
- inputs combinations sufficient for producing a certain quantity of output

Input requirement set



Open and closed set

- The open ε-ball with center $\mathbf{x}^0$ and radius $\varepsilon > 0$ is a subset of points in $\mathbb{R}^n$:

$$B_\varepsilon(\mathbf{x}^0) \equiv \{\mathbf{x} \in \mathbb{R}^n | \, d(\mathbf{x}^0, \mathbf{x}) < \varepsilon\}$$

- The closed ε-ball:

$$B_\varepsilon(\mathbf{x}^0) \equiv \{\mathbf{x} \in \mathbb{R}^n | \, d(\mathbf{x}^0, \mathbf{x}) \leq \varepsilon\}$$

Open and closed set

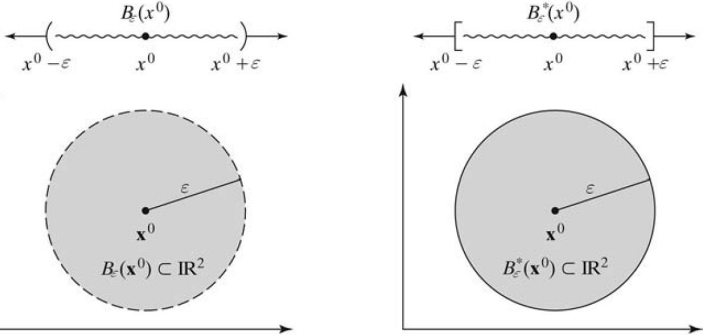


Figure A1. 10. Balls in $\mathbb{R}$ and $\mathbb{R}^2$.

Source: Jehle & Reny (2011)

Open and closed set

- $S \subset \mathbb{R}^n$ is an **open set** if for all $\mathbf{x} \in S$, there exists some $\varepsilon > 0$ such that $B_\varepsilon(\mathbf{x}) \subset S$.
- S is a **closed set** if its complement S^c is an open set.

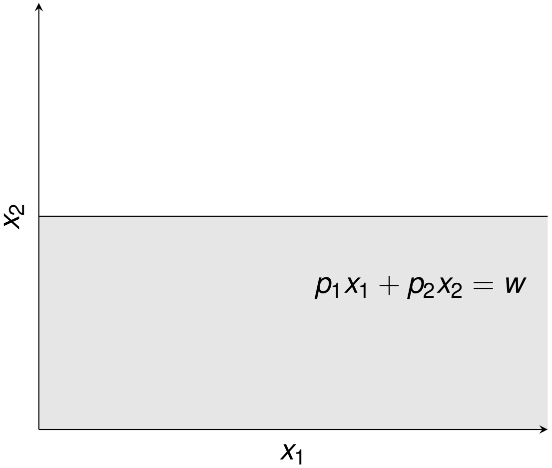
Open and closed set

Question: Are these sets open or closed?

- $\emptyset$
- $\mathbb{R}^n$
- the union of open sets
- the intersection of any finite number of open sets
- the union of any finite number of closed sets
- the intersection of closed set
- the intersection of a closed set and an open set

Bounded set

A set $S \subset \mathbb{R}^n$ is **bounded** if it is entirely contained with some ε-ball (either open or closed).

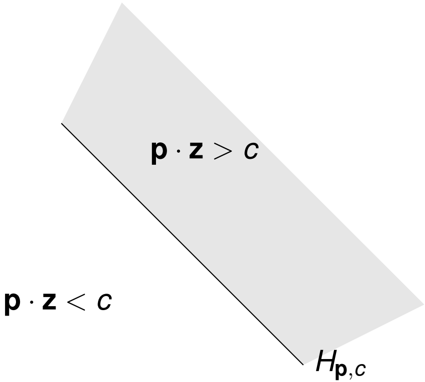


Compact set

A set $S \subset \mathbb{R}^n$ is **compact** if it is both closed and bounded.

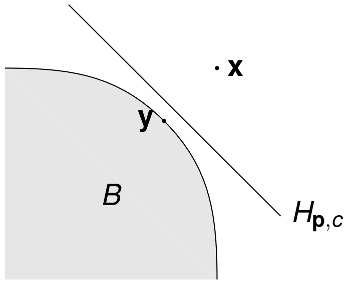
Separating hyperplane theorem

Given $\mathbf{p} \in \mathbb{R}^n$ with $p \neq 0$ and $c \in \mathbb{R}$, the **hyperplane** generated is the set $H_{\mathbf{p},c} = \{z \in \mathbb{R}^n | \mathbf{p} \cdot \mathbf{z} = c\}$



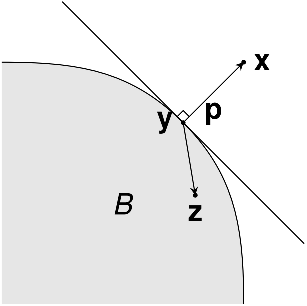
Separating hyperplane theorem

Suppose the $B \subset \mathbb{R}^n$ is a convex and closed set and that $\mathbf{x} \notin B$. Then there is $\mathbf{p} \in \mathbb{R}^n$ and a value $c \in \mathbb{R}$ such that $\mathbf{p} \cdot \mathbf{x} > c$ and $\mathbf{p} \cdot \mathbf{y} < c$ for every $\mathbf{y} \in B$



It is used to prove the Second Welfare theorem, which implies for any initial endowment distribution, there is a price set that supports a redistribution of endowments toward a Pareto optimal in an exchange economy.

Separating hyperplane theorem



Proof:

1. We can find a point $\mathbf{y} \in B$ that is closest to the $\mathbf{x} \notin B$.
2. Denote $\mathbf{p} = \mathbf{x} - \mathbf{y}$ and $c' = \mathbf{p} * \mathbf{y}$.
3. $\mathbf{px} - \mathbf{c}' = \mathbf{px} - \mathbf{py} = (\mathbf{x} - \mathbf{y})^2 > 0$.
4. For any $\mathbf{z} \in B$, $\mathbf{p} * (\mathbf{z} - \mathbf{y}) = \mathbf{pz} - c' \leq 0$ because vector $\mathbf{p}$ and $\mathbf{z} - \mathbf{y}$ cannot make an acute angle.
5. $\mathbf{px} > c'$ and $\mathbf{pz} \leq c' \implies \exists \varepsilon \to 0$ such that $\mathbf{p} * \mathbf{x} > c$ and $\mathbf{p} * \mathbf{y} < c$ for $c = c' + \varepsilon$.